Sage computations of $\mathfrak{sl}_2(k)$-Levi extensions

Pilar Benito*, Daniel de-la-Concepción

Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio Vives, Luis de Ulloa s/n, 26004 Logroño, Spain.

E-mail: pilar.benito@unirioja.es, daniel-de-la-concepcion@alum.unirioja.es

Abstract

In 2010, Snobl [9] studied the structure of nilpotent Lie algebras admitting a Levi extension. As a corollary of the results therein, it is shown that the classes of characteristically nilpotent or filiform Lie algebras do not admit Levi extensions. The paper ends by asking for the possibility of finding series of nilpotent Lie algebras in arbitrary dimension not being abelian or Heisenberg and allowing such extensions. Our goal in this work is to present computational examples of this type of algebras by using Sage software. In the case of nilpotent Lie algebras admitting $\mathfrak{sl}_2(k)$ as Levi factor special constructions will be given by means of Sage routines based on transvections over $\mathfrak{sl}_2(k)$-irreducible modules.

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1 Introduction

Levi's theorem (Eugenio Elia Levi, 1905 [8]) decomposes any arbitrary Lie algebra of characteristic zero as a direct sum of a semisimple Lie algebra, known as *Levi factor*, and its solvable radical. Given a solvable Lie algebra R, a semisimple Lie algebra S is said to be a *Levi extension* of R if a Lie structure can be defined on the vector space $S \oplus R$. The assertion is equivalent to $\rho(S) \subseteq \mathrm{Der}(R)$, where $\mathrm{Der}(R)$ is the derivation algebra of R, for some representation ρ of S onto R. Since $\rho(S)$ is a semisimple Lie algebra, then a Lie algebra with solvable derivation algebra has no Levi extensions. This is the case of the class of characteristically nilpotent Lie algebras introduced by Dixmier and Lister in [3].

In 2010, Snobl [9] studied the structure of nilpotent Lie algebras admitting a Levi extension and established the following preliminary result on their structure:

Theorem 1.1 (Snobl, Theorem 2 [9])**.** Let L be an indecomposable Lie algebra with product $[x, y]$, nilpotent radical N of $(t+1)$-nilindex and a nontrivial Levi decomposition $L = N \oplus S$ for some semisimple Lie algebra S. Then, there exists a decomposition of N into a sum of $ad\, S$-modules:

$$N = m_1 \oplus m_2 \oplus \cdots \oplus m_t$$

where

$$N^j = m_j \oplus N^{j+1}, \quad m_j \subseteq [m_{j-1}, m_j]$$

such that m_1 is a faithful $ad\, S$-module and for $2 \leq j \leq t$, the $ad\, S$-module m_j decomposes into a sum of some subset of irreducible components of the tensor representation $m_1 \otimes m_{j-1}$. $\diamondsuit$

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In [9], Snobl asked about the possibility of finding series of nilradicals in arbitrary dimension, other than abelian or Heisenberg, allowing Levi extensions. This is the main goal of this paper. Starting from free nilpotent Lie algebras (all of them admit Levi extensions) and using Sage we will display some computational examples of nilpotent Lie algebras admitting the 3-dimensional simple split Lie algebra $\mathfrak{sl}_2(k)$ as Levi factor.

The paper is organised into four sections apart from the Introduction. In Section 2 we introduce the main definitions and results we will need throughout the paper. Section 3 reviews the representation theory of $\mathfrak{sl}_2(k)$ and revisits the main tool of transvections which allows us to build Sage routines in Section 4 for computing nilpotent Lie algebras. The paper ends with a final section of examples.

Vector spaces in this paper are considered to be finite-dimensional over a field $\mathbb{K}$ of characteristic zero. The non defined concepts and basic statements can be founded in [6], [7] and [1].

2 Preliminary concepts and results

First, let us take a look at the results to be used in the implementation of the Sage routines. Most of them are elementary and well known for people working on Lie algebras. In this case, the section may be dropped.

A *Lie algebra* L is a vector space endowed with a binary skew-symmetric bilinear product $[a, b]$ satisfying the Jacobi identity:

$$J(a, b, c) = [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0. \tag{1.1}$$

The Lie bracket of two vector subspaces U, V of L is defined as the whole span

$$[U, V] = span < [u, v] : u \in U, v \in V > .$$

For a given associative algebra $(A, a \cdot b)$, the vector space A with the skew-symmetric product $[a, b] = a \cdot b - b \cdot a$ is a Lie algebra; this algebra is denoted as A^-. In fact, from the Ado-Ivasawa's theorem, any Lie algebra is just isomorphic to a subalgebra of A^- for some associative algebra A. The main example of the Lie algebra A^- is the *general linear Lie algebra* $\mathfrak{gl}(V)$ defined from the associative algebra of endomorphisms over a vector space V, i.e. $\mathfrak{gl}(V) = End(V)^-$ (once a basis of V is fixed, $\mathfrak{gl}(V)$ can be viewed as $\mathcal{M}_n(k)^-$, where n is the dimension of V).

A *simple* Lie algebra is a non abelian Lie algebra without proper ideals. The Lie algebras which are direct sum of simple ideals are called *semisimple*. Throughout the paper, we will use the following series of ideals of the Lie algebra L. The *derived series* is defined recursively as $L = L^{(1)}$ and $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$ for $n > 1$; the *lower central series* is again defined recursively as $L = L^1$ and $L^n = [L, L^{n-1}]$. If the derived series vanishes, i.e. there exists $k \in \mathbb{N}$ such that $L^{(k)} = 0$, then L is called *solvable*. If the lower central series vanishes, then L is *nilpotent*. The smallest value of k for which $L^k = 0$ is called *degree of nilpotency or nilindex* of L. The solvable radical of L, denoted as $R(L)$, is the biggest solvable ideal of L. Also the *nilpotent radical* $N(L)$ of L, can be defined as the largest nilpotent ideal of L. From Levi's Theorem, any Lie algebra can be built from solvable and semisimple ones:

Theorem 2.1 (Eugenio E. Levi, 1905 [8])**.** Given L a finite-dimensional Lie algebra of characteristic zero, then there exists a semisimple Lie algebra $S \leq L$ such that $L = S \oplus R(L)$. $\diamondsuit$

The adjoint map $ad : S \to \mathfrak{gl}(R(L))$ given by $adx(a) = [x, a]$ is an homomorphism of Lie algebras and allows us to see the radical of a Lie algebra L as an S-module. In general,

Definition 2.2. A *representation* of an arbitrary Lie algebra L is an homomorphism of Lie algebras

$$\rho : L \to \mathfrak{gl}(V)$$

where V is a vector space. The vector space V and the action $x \cdot v = \rho(x)(v)$ is called *L-module.* The module V is called *irreducible* if it is non-trivial and does not contain proper submodules.

Lemma 2.3. If V and W are L-modules, then $V \otimes W$ is L-module under the action given by:

$$g \cdot (v \otimes w) = (g \cdot v) \otimes w + v \otimes (g \cdot w).$$

Moreover, for any $n \geq 2$, the vector spaces $S^n(V)$ and $\Lambda^n(V)$, of n-power symmetric and skew-symmetric vectors respectively, are submodules of the n-power tensor product module $\otimes^n V$. $\diamondsuit$

Throughout the paper, we will build Lie algebras with nilpotent solvable radical (i.e., $R(L) = N(L)$) by gluing a nilpotent Lie algebra N and the simple Lie algebra $\mathfrak{sl}_2(k)$, which is the subalgebra of traceless 2×2 matrices of $\mathcal{M}_2(k)^-$. This can be done by using representations $\rho : \mathfrak{sl}_2(k) \to \mathfrak{gl}(N)$ such that $\rho(\mathfrak{sl}_2(k))$ is contained in the *Lie algebra of derivations* of N,

$$Der\ N = \{d \in End(N) : d([x, y]) = [d(x), y] + [x, d(y)]\}.$$

Free nilpotent Lie algebras: Given a finite set $X = \{x_1, \dots, x_m\}$ and the vector space V with basis X, the free associative algebra generated by X can be defined as:

$$\mathfrak{F} = k1 \oplus \sum_{j \geq 1}^{\infty} \otimes^j V$$

We rewrite $V_j = \otimes^j V$ and $V^j = \Sigma_{k \geq j} V_j$. In [7] it is said that the free Lie algebra generated by X, and denoted by $\mathfrak{FL}$, is the subalgebra of $(\mathfrak{F}^-, [ab] = ab - ba)$ generated by X. Then,

$$\mathfrak{FL} = \sum_{j \geq 1} \mathfrak{FL}_j$$

where $\mathfrak{FL}_j$ is the subspace generated by the linear combinations of elements of the form $[x_{i_1}, \dots, x_{i_j}] = [\dots [x_{i_1} x_{i_2}] \dots x_{i_j}]$, with $x_{i_s} \in X$. For any $k \geq 2$, denoting by $\mathfrak{FL}^k$ the ideal $\sum_{j \geq k} \mathfrak{FL}_j$, we consider the quotient Lie algebra:

$$N(m, k) = \mathfrak{FL} / \mathfrak{FL}^{k+1}$$

Proposition 2.4 (Proposition 1.4 in [5])**.** The Lie algebra $N(m, k)$ is nilpotent of $(k+1)$-nilindex and is generated by m elements as subalgebra. Even more, any nilpotent Lie algebra of $(k+1)$-nilindex being generated by m elements is a homomorphic image of $N(m, k)$. $\diamondsuit$

The algebra $N(m, k)$ is called *free nilpotent Lie algebra of type* $m = \mid X \mid$ *and* $(k+1)$*-nilindex.*

By considering modules, if S is a (semisimple) Lie algebra and $\mathfrak{FL}$ is generated by an S-module $V = span < X >$, the free Lie algebras $\mathfrak{FL}$ and $N(m, k)$ for all $k \in \mathbb{N}$ are also S-modules. We will denote by $N(V,k)$ the Lie algebra $N(\dim(V),k)$ but with the module structure induced by V in the natural way suggested by Lemma 2.3. The Lie algebras $N(V,k)$ and $N(W,k)$ may be isomorphic as algebras, but not as S-modules.

From [5], $N(V,2) = V \oplus \Lambda^2 V$ and it is not difficult to see that $N(V,3) = V \oplus \Lambda^2 V \oplus S^{(2,1)}V$ where, given a partition λ, $S^\lambda V$ is the irreducible $GL(V)$-submodule of $V^{\otimes|\lambda|}$ associated with the partition λ.

3 $\mathfrak{sl}_2(k)$-modules and transvections

The Lie algebra $\mathfrak{sl}_2(k)$ of 2×2 traceless matrices is a 3-dimensional simple Lie algebra. It is usually described by means of its standard basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.2}$$

which provides the law: $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$. This Lie algebra has a very interesting representation theory and a very simple arithmetic. Following [6], for each $n \in \mathbb{N}$ there exists a unique $(n+1)$-dimensional representation up to isomorphisms.

Let us consider $V(n)$ as the $\mathbb{K}$-vector space of the homogeneous polynomials of degree n in the variables x and y. Then $\mathfrak{sl}_2(k)$ acts on $V(n)$ in a natural way once $\mathfrak{sl}_2(k)$ is identified as derivations in the following way:

$$\begin{array}{ccc} \mathfrak{sl}_2(k) & \to & End(V(n)) \\ h & \mapsto & x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y} \\ e & \mapsto & x\dfrac{\partial}{\partial y} \\ f & \mapsto & y\dfrac{\partial}{\partial x} \end{array}$$

For any $n \in \mathbb{N}$, the previous identification defines an $\mathfrak{sl}_2(k)$-irreducible representation. In fact, these are all the finite-dimensional irreducible representations of $\mathfrak{sl}_2(k)$. Moreover, the polinomial set

$$\left\{ v_i = \binom{n}{i} x^{n-i} y^i : i = 0, \dots n \right\} \tag{1.3}$$

works as the standard basis built in [6] for the $(n+1)$-dimensional $\mathfrak{sl}_2(k)$-irreducible representation.

On the other hand, the Clebsch-Gordan's formula gives a decomposition of the tensor product of two irreducible representations as direct sum of its irreducible components:

$$V(n) \otimes_k V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \dots \oplus V(n-m) \tag{1.4}$$

under the assumption $n \geq m$. The Schur's lemma, over algebraically closed fileds, and irreducible modules, stablishes that

$$dim\ Hom_S(V,W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

Applying this result to $S = \mathfrak{sl}_2(k)$ and the tensor product $V(n) \otimes_k V(m)$ we get that $Hom_S(V(n) \otimes_k V(m), V(n+m-2q))$ is a 1-dimensional vector space for $q \leq m$. In conclusion, all the homomorphisms are scalar multiples of each other. So, we only need to find one non-zero homomorphism.

Let us consider the mapping introduced by Dixmier [4] as q-transvection, $(\cdot,\cdot)_q : V(n)\times V(m) \to V(n+m-2q)$. For $f \in V(n)$ and $g \in V(m)$, the *transvector* $(f,g)_q$ is defined by:

$$(f,g)_q = \frac{(m-q)!}{m!}\frac{(n-q)!}{n!}\sum_{i=0}^{q}(-1)^i\binom{q}{i}\frac{\partial^q f}{\partial x^{q-i}\partial y^i}\frac{\partial^q g}{\partial x^i \partial y^{q-i}} \tag{1.5}$$

Since derivations are linear applications, transvections are bilinear maps. Even more, as $\mathfrak{sl}_2(k)$ acts as derivations on $V(n)$ and $V(m)$, transvections are nonzero homomorphism of $\mathfrak{sl}_2(k)$-modules as it is explained in [4]. So:

$$Hom_S(V(n)\otimes_k V(m), V(n+m-2q)) = span < (\cdot,\cdot)_q >$$

Now, the ideas of our two preliminary sections can be applied to $\mathfrak{sl}_2(k)$. Let us consider the nilpotent Lie algebra $N(V(n),2)$. Its expansion is $m_1 = V(n)$ and if n is odd $m_2 = \bigwedge^2 V(n) = V(2n-2)\oplus V(2n-6)\oplus\ldots\oplus V(0)$; but if n is even $m_2 = \bigwedge^2 V(n) = V(2n-2)\oplus V(2n-6)\oplus\ldots\oplus V(2)$. The product of this Lie algebra is given, up to isomorphisms, by the formula:

$$[v,w] = (v,w)_1 + (v,w)_3 + \ldots + (v,w)_n \text{ if } n \text{ is odd},$$

$$[v,w] = (v,w)_1 + (v,w)_3 + \ldots + (v,w)_{n-1} \text{ if } n \text{ is even}.$$

The Lie algebra $N(V(n),3)$ is not as simple to describe as $N(V(n),2)$. The expansion of the module $m_3 = S^{(2,1)}V(n)$ is not easy in a general setting, but once n is fixed there are several combinatorial mechanisms to compute it. There are even computer software specifically designed to do these computations as fast as possible.

In the next section, we will give routines in Sage which lead to compute multiplication tables of nilpotent Lie algebras admitting $\mathfrak{sl}_2(k)$ as Levi factor (equivalently, nilpotent Lie algebras N such that $\mathfrak{sl}_2(k) \subseteq \mathrm{Der}\, N$). These algebras turn to be quotients of $N(V(n),k)$. As in [2] and for simplicity, we will consider a Lie algebra L whose ideals are in a chain. The Levi decomposition of L is $L = S\oplus N$, where S is a simple Lie algebra, N a nilpotent one and the quotient N^j/N^{j+1} of two consecutively terms in the lower central series of N is a S-irreducible module with N/N^2 being nontrivial. If N is of 3-nilindex and $S = \mathfrak{sl}_2(k)$, N was completely described in [2] as $N = m_1\oplus m_2$ where $m_1 = V(n)$ and $m_2 = V(2n-2k)$ with k being odd; a complete product could be given in a recursive way for N, by taking the standard basis in Eq. (1.3) of the irreducible module m_i. The 4-nilindex case is given by the decomposition $N = m_1\oplus m_2\oplus m_3$ where $m_1 = V(n)$, $m_2 = V(2n-2k)$ taken k being odd and $m_3 = V(3n-2k-2q)$, $q \le \min\{n, 2n-2k\}$, with m_3 being an irreducible submodule of $S^{(2,1)}V(n)$ with some additional restrictions (see [1]).

4 Source code with Sage

From previous concepts, results and remarks and using transvections, the standard basis, $\{h,e,f\}$ of $\mathfrak{sl}_2(k)$ given in Eq. (1.2) and the basis in Eq. (1.3) for $V(n)$ as main tools, we will display several Sage implementations.

`Transvection Implementation:` The next routine implements the transvection $(\ ,\ )_k : V(n)\otimes V(m) \to V(n+m-k)$ as it is defined in Eq. (1.5) in the following way:

$$(v_i, v_j)_k = prod(i,j,n,m,k)v_{i+j-2k}.$$

```
def prod(i,j,n,m,k):
    res=0;
    if (i+j<k):
        return res;
    if (n+m<i+j+k):
        return res;
    for t in range(k+1):
        res=res+(-1)^t*binomial(n-k,i-t)*
binomial(m-k,m-j-t)*binomial(k,t);
    res=res/binomial(n+m-2*k,i+j-k);
    return res;
```

By using the previous implementation of the transvection map, we can implement all types of diferent multiplication tables of nilpotent Lie algebras that are modules for $\mathfrak{sl}_2(k)$. In particular we will implement a routine for 3- and 4-nilpotent Lie algebras which fit as the radical (in fact nilradical) of Lie algebras whose ideals are in a chain. In createTable2(n,k,r), if $V(3n-2k-2r)$ is not an appropiate submodule of $S^{(2,1)}V(n)$, then the matrix given is not the multiplication table of a Lie algebra. It has to be previously checked according to the results obtained in [1] or tested subsequently by using the Jacobi identity in Eq. (1.1). We present here a short Sage program for testing if an introduced multiplication table satisfies the Jacobi identity:

`Jacobi identity checking Program`: Algorithm to check if a skew symmetric multiplication table corresponds to a Lie algebra:

```
def CheckJacobi(n,k,r):
    m=2*(n-k);
    for i in range(n+1):
     for j in range(i+1,n+1):
     for t in range (j+1,n+1):
     if (prod(i,j,n,n,k)*prod(i+j-k,t,m,n,r)+prod(j,t,n,n,k)*
prod(t+j-k,i,m,n,r)+prod(t,i,n,n,k)*prod(i+t-k,j,m,n,r)!=0):
                print "The Jacobi identity is not fulfilled";
                return;
    print "The Jacobi identity is fulfilled";
```

`3-nilpotent Program`: Implementation of the routine which provides multiplication tables for a 3-nilpotent Lie algebra

$$N_{n,k} = V(n) \oplus V(2n-2k).$$

```
def createTable(n,k):
    m=2*(n-k);
    V=QQ^(n+1+m+1);
    VB=V.basis();
    A=[];
```

```
    B=[];
    C=[];
    D=[];
    v = list(var('v_%d' % i) for i in range(n+1));
    w = list(var('w_%d' % i) for i in range(m+1));
    variables=v+w;
    M=PolynomialRing(QQ,variables);
    variables=M.gens();
    pol=M.zero();
    B=[];
    for s in range(k):
        B.append(pol);
    for j in range(k,n+1):
        B.append(binomial(m+k-j,n-j)/
binomial(m,n-k)*variables[n+1+j-k]);
    for s in range(n+1,m+1+n+1):
        B.append(pol);
    A.append(B);
    B=[];
    pol=M.zero();
    for t in range(1,n+1):
        for s in range(t+1):
            B.append(pol);
        for s in range(t+1,n+1):
            cal=prod(t,s,n,n,k);
            if (cal!=0):
                 B.append(prod(t,s,n,n,k)*
variables[n+1+s+t-k]);
            else:
                B.append(pol);
        for s in range(n+1,n+1+m+1):
            B.append(pol);
        pol=M.0-M.0;
        A.append(B);
        B=[];
    for t in range(n+1,n+1+m+1):
        for s in range (n+1+m+1):
            B.append(M.zero());
        A.append(B);
        B=[];
    for s in range(0,n+1+m+1):
        for t in range(s):
            A[s][t]=-A[t][s];
    return matrix(A);
```

`4-nilpotent Program:` Implementation of the routine which provides multiplication tables for a 4-nilpotent algebra

$$N_{n,k,r} = V(n) \oplus V(2n-2k) \oplus V(3n-2k-2r).$$

```
def createTable2(n,k,r):
    m=2*(n-k);
    g=n+m-2*r;
    V=QQ^(n+1+m+1+g+1);
    VB=V.basis();
    A=[];
    B=[];
    C=[];
    D=[];
    v = list(var('v_%d' % i) for i in range(n+1));
    w = list(var('w_%d' % i) for i in range(m+1));
    z = list(var('z_%d' % i) for i in range(g+1));
    variables=v+w+z;
    M=PolynomialRing(QQ,variables);
    variables=M.gens();
    pol=M.zero();
    B=[];
    for s in range(k):
        B.append(pol);
    for j in range(k,n+1):
        B.append(binomial(m+k-j,n-j)/
binomial(m,n-k)*variables[n+1+j-k]);
    for s in range(n+1,m+1+n+1):
        B.append(prod(0,s-n-1,n,m,r)*variables[m+1+s-r]);
    for s in range(m+1+n+1,m+1+n+1+g+1):
        B.append(pol);
    A.append(B);
    B=[];
    pol=M.zero();
    for t in range(1,n+1):
        for s in range(t+1):
            B.append(pol);
        for s in range(t+1,n+1):
            B.append(prod(t,s,n,n,k)*variables[n+1+s+t-k]);
        for s in range(n+1,n+1+m+1):
            cal=prod(t,s-n-1,n,m,r);
            if (cal!=0):
                B.append(prod(t,s-n-1,n,m,r)*
variables[m+1+s+t-r]);
            else:
                B.append(pol);
```

```
        pol=M.0-M.0;
        for s in range(n+1+m+1,m+1+n+1+g+1):
            B.append(pol);
        A.append(B);
        B=[];
    for t in range(n+1,n+1+m+1+g+1):
        for s in range (n+1+m+1+g+1):
            B.append(M.zero());
        A.append(B);
        B=[];
    for s in range(0,n+1+m+1+g+1):
        for t in range(s):
            A[s][t]=-A[t][s];
    return matrix(A);
```

`4-ideal Lie algebras Program:` Implementation of the routine which provides multiplication tables for a Lie algebra

$$\mathcal{L}(n,k) = \mathfrak{sl}_2(k) \oplus V(n) \oplus V(2n-2k)$$

following [2]. These type of algebras have exactly 4 ideals.

```
def multY(w,n,variables,M):
    Z=parent(w);
    z=M.0;
    z=z-z;
    q=Z.dimension();
    m=q-5-n;
    if w[1]!=0:
        z=z-w[1]*M.0;
    if w[0]!=0:
        z=z+2*w[0]*M.2;
    for j in range(3+n+1,q-1):
       if w[j] != 0:
           z=z + (w[j]*(j-3-n))*variables[j+1];
    for j in range(3,3+n):
       if w[j] != 0:
           z=z + (w[j]*(j+1-3))*variables[j+1];
    return z;

def multX(w,n,variables,M):
    Z=parent(w);
    z=M.0;
    z=z-z;
    q=Z.dimension();
    m=q-5-n;
```

```
    if w[2]!=0:
        z=z+w[2]*M.0;
    if w[0]!=0:
        z=z-2*w[0]*M.1;
    for j in range(3+n+2,q):
       if w[j] != 0:
           z=z + (w[j]*(m-j+1+3+n+1))*variables[j-1];
    for j in range(3+1,3+n+1):
       if w[j] != 0:
           z=z + (w[j]*(n-j+1+3))*variables[j-1];
    return z;

def multH(w,n,variables,M):
    Z=parent(w);
    z=M.0;
    z=z-z;
    q=Z.dimension();
    m=q-5-n;
    ZB=Z.basis();
    if w[1]!=0:
        z=z+w[1]*2*M.1;
    if w[2]!=0:
        z=z-2*w[2]*M.2;
    for j in range(3+n+1,q):
       if w[j] != 0:
           z=z + (w[j]*(m-2*(j-3-n-1)))*variables[j];
    for j in range(3,3+n+1):
       if w[j] != 0:
           z=z + (w[j]*(n-2*(j-3)))*variables[j];
    return z;

def ToVector(pol,n,k,variables):
    a=3+n+1+2*(n-k)+1;
    V=QQ^a;
    VB=V.basis();
    v=V.0-V.0;
    for t in range(a):
        d=dict(zip(variables,VB[t]))
        v[t]=pol.subs(d);
    return v;

def createTable3(n,k):
    m=2*(n-k);
    V=QQ^(3+n+1+m+1);
    VB=V.basis();
    A=[];
```

```
B=[];
C=[];
D=[];
sl=list(var('h x y'))
v = list(var('v_%d' % i) for i in range(n+1));
w = list(var('w_%d' % i) for i in range(m+1));
variables=sl+v+w;
M=PolynomialRing(QQ,variables);
variables=M.gens();
pol=M.zero();
for t in range(3+n+1+m+1):
    S=multH(VB[t],n,variables,M);
    B.append(S);
    S=multX(VB[t],n,variables,M);
    C.append(S);
    S=multY(VB[t],n,variables,M);
    D.append(S);
A.append(B);
A.append(C);A.append(D);
B=[];
pol=M.zero();
for s in range(3+k):
    B.append(pol);
for j in range(3+k,3+n+1):
    B.append(binomial(m+k-j+3,n-j+3)/binomial(m,n-k)*variables[n+1+j-k]);
for s in range(3+n+1,3+m+1+n+1):
    B.append(M.zero());
A.append(B);
B=[];
pol=M.zero();
for t in range(4,3+n+1):
    for s in range(t+1):
        B.append(pol);
    for s in range(t+1,3+n+1):
        if s<3+n:
            B.append(1/(t-3)*multY(ToVector(A[t-1][s],n,k,
            variables),n,variables,M)-(s-2)/(t-3)*(A[t-1][s+1]));
        else:
            B.append(1/(t-3)*multY(ToVector(A[t-1][s],n,k,
            variables),n,variables,M));
        pol=M.0-M.0;
    for s in range(3+n+1,3+n+1+m+1):
        B.append(pol);
    A.append(B);
    B=[];
```

```
    for t in range(3+n+1,3+n+1+m+1):
        for s in range (3+n+1+m+1):
            B.append(M.zero());
        A.append(B);
        B=[];
    for s in range(3,3+n+1+m+1):
        for t in range(s):
            A[s][t]=-A[t][s];
    return matrix(A);
```

5 Examples

Table 1 displays the decomposition as $\mathfrak{sl}_2(k)$-irreducible modules of the free nilpotent Lie algebras of type $n+1$ and 3- and 4-nilindex being generated by the irreducible $\mathfrak{sl}_2(k)$-modules $V(n)$ for $n = 1, 2, 3$. Then, $N(n+1, 2) = N(V(n), 2) = V(n) \oplus \Lambda^2 V(n)$, $N(n+1, 3) = N(V(n), 3) = V(n) \oplus \Lambda^2 V(n) \oplus S^{(2,1)} V(n)$, the spaces $\Lambda^2 V(n)$ and $S^{(2,1)} V(n)$ inherit the module structure from $V(n)$. The module decompositions in the table are obtained following Clebsch-Gordan's formula in Eq. (1.4). From this information and the previous Sage routines, we will get multiplication tables of some nilpotent Lie algebras admitting $\mathfrak{sl}_2(k)$ as Levi factor. All the nilpotent algebras given in the examples satisfy that quotients of two consecutively terms in their lower central series (l. c. s. in the sequel) are $\mathfrak{sl}_2(k)$-irreducible modules.

Table 1. $\mathfrak{sl}_2(k)$-free nilpotent $N(V(n), 2)$ and $N(V(n), 3)$

n	V	$\bigwedge^2 V$	$S^{(2,1)} V$
1	$V(1)$	$V(0)$	$V(1)$
2	$V(2)$	$V(2)$	$V(2) \oplus V(4)$
3	$V(3)$	$V(0) \oplus V(4)$	$V(1) \oplus V(3) \oplus V(5) \oplus V(7)$
4	$V(4)$	$V(2) \oplus V(6)$	$V(2) \oplus 2V(4) \oplus V(6) \oplus V(8) \oplus V(10)$

Example 5.1. The free nilpotent Lie algebra $N(V(1), 2)$ of type 2 and 3-nilindex is just:

$$N_{1,1} = V(1) \oplus k = span < v_0, v_1 > \oplus < w_0 > .$$

The nonzero multiplication table from Sage is given by:

$$[v_0, v_1] = w_0.$$

This algebra is nothing else but the Heisenbreg 3-dimensional Lie algebra. The free nilpotent Lie algebra $N(V(1), 3)$ of type 2 and 4-nilindex can be described as the 5-dimensional algebra:

$$N_{1,1,0} = V(1) \oplus k \oplus V(1) = \mathrm{span}\langle v_0, v_1\rangle \oplus \mathrm{span}\langle w_0\rangle \oplus \mathrm{span}\langle z_0, z_1\rangle$$

with nonzero multiplication table given by:

$$[v_0, v_1] = w_0,$$

$$[v_0, w_0] = z_0,$$
$$[v_1, w_0] = z_1.$$

These algebras are the unique nilpotent Lie algebras of 3- and 4-nilindex such that the codimension of its derived algebra N^2 equals 2. $\diamondsuit$

Example 5.2. The free nilpotent Lie algebra $N(V(2), 2)$ of type 3 and 3-nilindex can be described as the 6-dimensional algebra:

$$N_{2,1} = V(2) \oplus V(2) = \text{span}\langle v_0, v_1, v_2\rangle \oplus \text{span}\langle w_0, w_1, w_2\rangle$$

In this case, the nonzero multiplication table provided by Sage is:

$$[v_0, v_1] = w_0,$$
$$[v_0, v_2] = \frac{1}{2}w_1,$$
$$[v_1, v_2] = w_2.$$

The 3rd term in the l. c. s. of the free nilpotent algebra $N(V(2), 3)$ decomposes as $V(2) \oplus V(4)$. Sage routines in Section 4, let us create nilpotent 4-nilindex Lie algebras with the following $\mathfrak{sl}_2(k)$-irreducible decomposition: the 9-dimensional $N_{2,2,1} = V(2) \oplus V(2) \oplus V(2)$ and the 11-dimensional $N_{2,2,0} = V(2) \oplus V(2) \oplus V(4)$. $\diamondsuit$

Example 5.3. The 2nd term in the l.c.s. of the free nilpotent Lie algebra $N(V(3), 2)$ of type 4 and 3-nilindex decomposes as $V(0) \oplus V(4)$. From Sage routines in Section 4, it is possible to build nilpotent 3-nilindex Lie algebras with the following $\mathfrak{sl}_2(k)$-irreducible decomposition: $N_{3,3} = V(3) \oplus V(0)$, which is the 5-dimensional Heisenberg algebra and the 9-dimensional Lie algebra:

$$N_{3,1} = V(3) \oplus V(4) = \text{span}\langle v_0, v_1, v_2, v_3\rangle \oplus \text{span}\langle w_0, w_1, w_2, w_3, w_4\rangle$$

with nonzero products given by

$$[v_0, v_1] = w_0,$$
$$[v_0, v_2] = \frac{1}{2}w_1,$$
$$[v_0, v_3] = \frac{1}{6}w_2,$$
$$[v_1, v_2] = \frac{1}{2}w_2,$$
$$[v_1, v_3] = \frac{1}{2}w_3,$$
$$[v_2, v_3] = w_4.$$

The 3rd term in the l.c.s. of $N(V(3), 3)$ decomposes as $V(1) \oplus V(3) \oplus V(5) \oplus V(7)$. In this case, any arbitrary recombination of the irreducible decomposition of $N(V(3), 3)$ does not work. In fact $N_{3,3,0} = V(3) \oplus V(0) \oplus V(3)$ does not provide a Lie algebra. So, as 2nd irreducible term, we can only choose $V(4)$ and the case $N_{3,1,2} = V(3) \oplus V(4) \oplus V(3)$ must be dropped. The decompositions

$N_{3,1,3} = V(3) \oplus V(4) \oplus V(1)$, $N_{3,1,1} = V(3) \oplus V(4) \oplus V(5)$ and $N_{3,1,0} = V(3) \oplus V(4) \oplus V(7)$ provide nilpotent Lie algebras. By using Sage, we have cheked the 15-dimensional Lie algebra $N_{3,1,1}$:

$$N_{3,1,1} = \mathrm{span}\langle v_0, v_1, v_2, v_3\rangle \oplus \mathrm{span}\langle w_0, w_1, w_2, w_3, w_4\rangle \oplus$$

$$\mathrm{span}\langle z_0, z_1, z_2, z_3, z_4, z_5\rangle$$

with nonzero multiplication table:

$$[v_0, w_1] = z_0;\ [v_0, w_2] = \frac{3}{5}z_1;\ [v_0, w_3] = \frac{3}{10}z_2,$$

$$[v_1, v_i] = \frac{1}{2}w_i,\ i > 1,$$

$$[v_1, w_0] = -z_0;\ [v_1, w_1] = -\frac{1}{5}z_1;\ [v_1, w_2] = \frac{3}{10}z_2;\ [v_1, w_3] = \frac{1}{2}z_3,$$

$$[v_1, w_4] = \frac{2}{5}z_4;\ [v_2, v_3] = w_4;\ [v_2, w_0] = -\frac{2}{5}z_1,$$

$$[v_2, w_1] = -\frac{1}{2}z_2;\ [v_2, w_2] = -\frac{3}{10}z_3;\ [v_2, w_3] = \frac{1}{5}z_4,$$

$$[v_2, w_4] = z_5;\ [v_3, w_0] = -\frac{1}{10}z_2;\ [v_3, w_1] = -\frac{3}{10}z_3,$$

$$[v_3, w_2] = -\frac{3}{5}z_4;\ [v_3, w_3] = -z_5. \quad \diamondsuit$$

Note that, as n grows, the tensor product decompositions increase the number of irreducible components in the free nilpotent Lie algebras. So, we can get lots of series of nilpotent Lie algebras admitting $\mathfrak{sl}_2(k)$ as Levi extensions.

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